Ph.D. Qualifying Exam: Algebra
August 2016

Student ID: Name:

Note: Be sure to use English for your answers.

1. [10 pts] Find the greatest common divisor of $8+6i$ and $5-15i$ in $\mathbb{Z}[i]$.
2. [15 pts] Let G be a finite group and suppose that G acts transitively on the finite set A. Let H be a normal subgroup of G and $B_1, B_2, \ldots, B_r$ be the distinct orbits of H on A. Prove that for each $g \in G$ and $i \in \{1, 2, \ldots, r\}$, there is a j such that $gB_i = B_j$
3. [15 pts] Let k be a field and X an independent variable over k. Show that $k(X)$ is a finite extension of $k\left(\frac{f(X)}{g(X)}\right)$ of degree $\max\{\deg f(X), \deg g(X)\}$.
4. [20 pts] Let E be a splitting field over F which is also a finite extension, and $f(X)$ be an irreducible polynomial in $F[X]$. Suppose that $g(X)$ and $h(X)$ are two irreducible factors of $f(X)$ in $E[X]$.
   (a) Show that there exists an automorphism $\sigma \in G(E/F)$ such that $g(X) = h^{\sigma}(X)$ where $h^{\sigma}(X) = \sigma(a_0) + \sigma(a_1)^X + \cdots + \sigma(a_r)X^r$ if $h(X) = a_0 + a_1X + \cdots + a_rX^r$.
   (b) Show by an example that the conclusion in (a) is false if E is not a splitting field over F.
5. [10 pts] Show that any finite subring of a division ring is a division ring.
6. [15 pts] Let A be a local ring with maximal ideal $\mathfrak{m}$ and M, N be finitely generated A-modules. Show that if $M \otimes N = 0$, then either $M = 0$ or $N = 0$.
7. [15 pts] Let R be a commutative ring with identity. Let I and J are ideals of R. Prove that $R/I \otimes_R R/J$ is isomorphic to $R/(I+J)$ as R-modules.

THE END

Ph.D. Qualifying Exam: Advanced Statistics

August 2016

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20 pts] Let the column vector $(X_1, X_2, X_3)'$ be a Normal random vector with distribution $N_3(\mu, \Sigma)$ where the mean is $\mu = (\mu_1, \mu_2, \mu_3)'$ and Σ is the covariance matrix whose value at entry (i, j) is σ_{ij}, for $i, j = 1, 2, 3$. Let the (1,3)- and (3,1)-entries of Σ^{-1} be equal to 0 and non-zero at the other entries.

   (a) Check if X_1 and X_3 are independent.

   (b) Check if X_1 and X_3 are conditionally independent given X_2.

   (c) Denote by ρ_{ij} the correlation coefficient between X_i and X_j. Express ρ_{13} in terms of ρ_{ij} except ρ_{13}.

2. [20 pts] Let $X_1, \cdots, X_n$ be iid $N(\theta, 1)$.

   (a) Find the conditional distribution of X_1 given $\sum_{i=1}^n X_i = t$.

   (b) Find the best unbiased estimator W of θ^2.

   (c) Find the variance of W.

   (d) Compare the variance of W with the Cramer-Rao lower bound.

3. [15 pts] Let X_1, X_2 be iid uniform$(\theta, \theta+1)$. For testing $H_0 : \theta = 0$ vs $H_1 : \theta > 0$, we have two competing tests, ϕ_1 and ϕ_2:

$$\begin{aligned} \phi_1(X_1) &: \text{Reject } H_0 \text{ if } X_1 > 0.95, \\ \phi_2(X_1, X_2) &: \text{Reject } H_0 \text{ if } X_1 + X_2 > C. \end{aligned}$$

   (a) Find the value of C so that test ϕ_2 has the same size as test ϕ_1.

   (b) Compute the power function of each test and draw the graph of each power function.

4. [15 pts] Prove that convergence in probability implies convergence in distribution.

5. [15 pts] Let $X_1, \cdots, X_n$ be a random sample from an $N(\theta, \theta^2)$ population, where $\theta > 0$.

   (a) Find a pivotal quantity based on the sample.

   (b) Find a $1 - \alpha$ confidence interval for θ using the pivotal quantity.

   (c) Check if $T = \sum_{i=1}^n X_i$ is complete.

6. [15 pts] Prove the following statement:
   Let there be a random sample from a distribution $f(x; \theta)$ and T be a statistic defined on the sample. Let T be complete and sufficient for θ, and let $\phi(T)$ be a function of T only. Then $\phi(T)$ is the best unbiased estimator of $E(\phi(T))$.

THE END

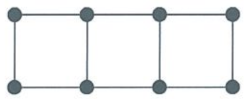
Ph.D. Qualifying Exam: Combinatorics
August 2016

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20 pts] Let $\mathcal{F}$ be a family of nonempty subsets of $\{1, 2, \ldots, n\}$. Suppose for any subsets $A, C \in \mathcal{F}$, if $A \subset B \subset C$, then $B \in \mathcal{F}$. Show that
$$\left|\sum_{A\in\mathcal{F}} (-1)^{|A|}\right| \le \binom{n}{\lfloor n/2 \rfloor}.$$

2. [20 pts] Find the chromatic polynomial of the following graph:



3. [20 pts] Let $\mathcal{A}$ be a subset of vectors in $\mathbb{Z}_3^n$. Suppose that for every pair of distinct vectors $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ in $\mathcal{A}$, there exists some coordinate $i \in \{1, 2, \ldots, n\}$ such that $u_i - v_i = 1 \pmod 3$. Show that $|\mathcal{A}| \le 2^n$.

4. [20 pts] Let $\mathcal{H}$ be a 3-uniform hypergraph with n vertices and αn^3 edges, where $n \ge \frac{1}{2\sqrt{\alpha}}$. Show that $\mathcal{H}$ has an independent set of vertices of size at least $\frac{3}{8\sqrt{\alpha}}$.

5. [20 pts] Let $\mathcal{C} = \binom{[17]}{\ge 2}$, that is, $\mathcal{C}$ is the family of subsets of $\{1, 2, \ldots, 17\}$ which contain at least two elements. Show that for any partition of $\mathcal{C}$ into 3 parts $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$, there exists distinct sets $A, B \in \mathcal{C}$, such that A, B, and $A \cup B$ belong to the same part $\mathcal{P}_i$.

THE END

Ph.D. Qualifying Exam: Algebraic Topology I
August 2016

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20 pts] Let $p : E \to B$ be the simply connected covering with E and B path-connected and locally path-connected. Let $T : B \to B$ be a map with a fixed point such that $T^m = \mathrm{Id}$. Show that there exists a map $\widetilde{T} : E \to E$ such that $p \circ \widetilde{T} = T \circ p$ and $\widetilde{T}^m = \mathrm{Id}$.

2. [10 pts each] Prove the following statements.
   (a) Let X be a topological space, and let $f, g : X \to S^n$ be any two continuous maps such that $f(x) \neq g(x)$ for every $x \in X$. Then f is homotopic to g.
   (b) Any continuous map $f : S^n \to S^1$ with $n \geq 2$ is homotopic to a constant map.

3. Let X be a path connected and locally path connected space such that $\pi_1(X)$ is finite. Let $Y = S^1 \times S^1 \times S^2$.
   (a) [5 pts] Find the universal covering of Y.
   (b) [15 pts] Show that any continuous map $f : X \to Y$ induces the trivial map $f_* : H_i(X : \mathbb{Z}) \to H_i(Y : \mathbb{Z})$ for all i different from 0 and 2.

4. Let X be a connected CW-complex with two 0-cells, three 1-cells, and three 2-cells. Assume that $H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}_3$.
   (a) [5 pts] Determine the Euler characteristic $\chi(X)$ of X.
   (b) [15 pts] Determine $H_2(X)$

5. [20 pts] For a topological space X, let ΣX denote the unreduced suspension of X. Using only the Eilenberg-Steenrod homology axioms (homotopy, exactness, excision, dimension) prove that $\widetilde{H}_i(\Sigma X)$ is naturally isomorphic to $\widetilde{H}_{i-1}(X)$. Here $\widetilde{H}_i$ denotes the reduced singular homology.

THE END

Ph.D. Qualifying Exam: Complex Analysis
August 2016

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] Show that if function f is holomorphic in a disk then f has a primitive in that disk.
2. [15 pts] State and give a proof of Casorati-Weierstrass Theorem.
3. [20 pts]
   (a) Show that the first terms of the Laurent series for $f(z) = \frac{\pi \csc \pi z}{z^2}$ are
   $$\frac{1}{z^3} + \frac{\pi^2}{6z} + \frac{7\pi^4 z}{360} + \ldots.$$
   (b) Determine the value of $\sum_{n=1}^{\infty} \frac{(-1)^2}{n^2}$ by integrating $f(z) = \frac{\pi \csc \pi z}{z^2}$ on an appropriate domain of the complex plane.
4. [15 pts] Suppose that f is holomorphic on a closed domain Ω, whose boundary is a simple closed curve γ. Suppose that f has n zeros $\{p_k\}$, $k = 1, \ldots, n$, in the interior of Ω and $f(z) \neq 0$ for $z \in \gamma$. If m_k is the multiplicity of the k-th zero $p_k \in \Omega$, show that
   $$\frac{1}{2\pi i} \int_\gamma z \frac{f'}{f} dz = \sum_{k=1}^{n} m_k p_k.$$
5. [15 pts] Show that the map
   $$f(z) = \frac{1+z}{1-z}$$
   is conformal on the upper half-disk $D^+ := \{z = x + iy : |z| < 1, y > 0\}$ and that $f(D^+)$ is the first quadrant $\{w = u + iv : u > 0, v > 0\}$. Study also the behavior of f on the boundary of D^+.
6. [10 pts] Show that if an entire function f has bounded real part, then f must be constant.
7. [10 pts] Consider the function $f(z) = e^{-iz^2}$ on the domain
   $$S := \{x + iy \in C : x \geq 0, y \geq 0\}.$$
   Find the value for $|f(z)|$ when z is a point of the boundary of S. Find a set of points in the interior of S such that $|f(z)| \to +\infty$ for $|z| \to +\infty$. Does that contradict the maximum modulus principle ?

THE END

Ph.D. Qualifying Exam: Real Analysis
August 2016

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of Lebesgue measurable functions in $\mathbb{R}^d$. Assume that $f_k \to f$ pointwise almost everywhere on a Lebesgue measurable set $E \in \mathbb{R}^d$ with $m(E) < \infty$. Prove that, for any $\epsilon > 0$, there exists a closed set $A_\epsilon \subset E$ such that $m(E \setminus A_\epsilon) < \epsilon$ and $f_k \to f$ uniformly on A_ϵ.

2. Compute the following with justification.

   (a) [10 pts] $\lim_{n\to\infty} \int_0^n (1+\frac{x}{n})^n e^{-x/2} dx$.

   (b) [10 pts] $\int_E ye^{-xy} \sin x \, dxdy$, where $E = (0,\infty) \times (0,1)$.

3. [15 pts] Let $f : \mathbb{R} \to \mathbb{R}$. Prove that f satisfies the Lipschitz condition

$$|f(x) - f(y)| \leq M|x - y|$$

   for some M and all $x, y \in \mathbb{R}$ if and only if f is absolutely continuous and $|f'(x)| \leq M$ for a.e. x.

4. For an operator on a Hilbert space $\mathcal{H}$, we say that T is an isometry if $\|Tf\| = \|f\|$ for all $f \in \mathcal{H}$.

   (a) [10 pts] Prove that $T^*T = I$ if T is an isometry.

   (b) [10 pts] Give an example of an isometry that is not unitary.

5. [15 pts] Assume that μ is a σ-finite measure on S. Suppose that $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. Prove that, for every $f \in L^p(S,\mu)$,

$$\|f\|_p = \sup\left\{\left|\int_S fg \, d\mu\right| : g \in L^q(X,\mu), \|g\|_q = 1\right\}.$$

6. [15 pts] Let ν be a signed measure. Prove that the total variation $|\nu|$ is a positive measure that satisfies $\nu \leq |\nu|$.

THE END

Ph.D. Qualifying Exam: Numerical Analysis
August 2016

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20 pts]
   (a) State Newton's method to find zeros of $f(x) = 0$ and prove the convergence. Consider various cases such as $f'(\alpha) = 0$ for some zero of f.
   (b) Generalize Newton's method to the system of nonlinear equations $\mathbf{f}(\mathbf{x}) = 0$. (No need to prove any theorem in this case)

2. [10 pts] Let $A, C \in \mathbb{C}^{n,n}$ and assume A is invertible with $\|A^{-1}\| \leq \alpha$. If $\|A - C\| \leq \beta$ and $\beta\alpha < 1$, then C is invertible and

$$\|C^{-1}\| \leq \frac{\alpha}{1 - \alpha\beta}.$$

3. [10 pts] Let $f \in C^2[a, b]$ and s be the natural cubic spline approximation of f with nodes $\{x_i\}_{i=0}^{n}$, $a = x_0$ and $b = x_n$ with $h = \max_k h_k$. Prove

$$\|f - s\|_\infty \leq h^{3/2} \left(\int_a^b |f''(x)|^2 dx \right)^{1/2}.$$

4. [10 pts] State Gaussian quadrature using n points for the approximation of integral $\int_a^b f(x)dx$ and prove it is exact for polynomials of degree $2n - 1$.

5. [20 pts] When we solve a system of equations numerically,

$$A\mathbf{x} = \mathbf{b}$$

we usually get approximate solution $\mathbf{x} + \mathbf{h}$. We would like analyze the relative error $\frac{\|\mathbf{h}\|}{\|\mathbf{x}\|}$ and find its relation with matrix A. We can view $\mathbf{x}+\mathbf{h}$ as the solution of a perturbed problem

$$(A + E)(\mathbf{x} + \mathbf{h}) = \mathbf{b} + \mathbf{k}.$$

Find a reasonable bound of $\frac{\|\mathbf{h}\|}{\|\mathbf{x}\|}$ in terms of the relative errors of $\|\mathbf{k}\|$ and $\|E\|$ under the assumption that the error $\|E\|$ is small enough.

6. [10 pts] State the Givens Householder algorithm to reduce an $n \times n$ real matrix A to an upper Hessenberg form.

7. [20 pts] State QR algorithm to find all the eigenvalues of A. Explain a detailed algorithm, computational complexity(i.e, the number of operations), effectiveness and efficiency, etc.

THE END

Ph.D. Qualifying Exam: Probability Theory
August 2016

Student ID: Name:

Note: Be sure to use English for your answers.

1. [10 pts] Let $\Omega = \{-1, 0, 1, 2\}$ and a function X on Ω be defined by $X(\omega) = \omega^2$ for $\omega \in \Omega$. Find the smallest σ-field $\sigma(X)$ that makes X a random variable.

2. [20 pts] Show the following statements.

   (a) [10 pts] Two random variables X_1 and X_2 are independent if and only if for every pair f_1, f_2 of non-negative bounded continuous functions, we have
   $$E[f_1(X_1)f_2(X_2)] = E[f_1(X_1)]E[f_2(X_2)].$$

   (b) [10 pts] Suppose for each n, that the pair ξ_n and η_n are independent random variables and that pointwise $\xi_n \to \xi_\infty$ and $\eta_n \to \eta_\infty$. Then the pair ξ_∞ and η_∞ are independent.

3. [10 pts] Show the following Kolmogorov Convergence Criterion: Suppose that $\{X_n, n \geq 1\}$ is a sequence of independent random variables. If $\sum_{i=1}^{\infty} Var[X_i] < \infty$, then $\sum_{i=1}^{\infty}(X_i - E[X_i])$ converges almost surely.

4. [15 pts] If X_n converges to X in L_p for $p \geq 1$, then show that $E[X_n^p]$ converges to $E[X^p]$.

5. [10 pts] If X_n converges in distribution to X_0 and $\sup_{n\geq 1} E[|X_n|^{2+\delta}] < \infty$ for some $\delta > 0$, then show that $\lim_{n\to\infty} E[X_n] = E[X_0]$ and $\lim_{n\to\infty} Var[X_n] = Var[X_0]$.

6. [10 pts] Let X, Y be random variables satisfying $X, XY \in L_1$. If $Y \in \mathcal{G}$, show that
   $$E[XY|\mathcal{G}] = YE[X|\mathcal{G}] \quad \text{almost surely.}$$

7. [25 pts] Let $\{\mathcal{F}_n, n \geq 1\}$ be an increasing sequence of σ-fields such that $\mathcal{F}_n \uparrow \mathcal{F}_\infty$. Consider a random variable X with $E[|X|] < \infty$.

   (a) [13 pts] Show that $\{E[X|\mathcal{F}_n], n \geq 1\}$ is a martingale with respect to $\{\mathcal{F}_n, n \geq 1\}$ and is uniformly integrable.

   (b) [12 pts] Show that $E[X|\mathcal{F}_n]$ converges to $E[X|\mathcal{F}_\infty]$ almost surely and in L_1.

THE END